Section 6.1

Friday, April 10, 2020 5:58 PM

Integration by parts

Recall that if $f(x)$ and $g(x)$ are differentiable, then

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \quad \text{and so} \quad f(x)g'(x) = (f(x)g(x))' - f'(x)\cdot g(x)$$

It follows that

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

and so

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$$

$$\boxed{\begin{array}{ll} u = f(x) & du = f'(x)\,dx \\ v = g(x) & dv = g'(x)\,dx \end{array}}$$

$$\boxed{\begin{array}{l} \int u\,dv = uv - \int v\,du \\ \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du \end{array}}$$

Example: Find $\int_0^1 x\sqrt{x+2}\,dx$ by integration by parts.

Use the substitution $w = x+2$ and integrate!

Solution:

$$\int_0^1 \underbrace{x}_{u}\underbrace{\sqrt{x+2}\,dx}_{dv} = x\cdot\frac{2}{3}(x+2)^{\frac{3}{2}}\Big|_0^1 - \int_0^1 \frac{2}{3}(x+2)^{\frac{3}{2}}\,dx = \left(\frac{2}{3}\cdot 3^{\frac{3}{2}} - 0\right) - \left(\frac{2}{3}\frac{(x+2)^{\frac{5}{2}}}{5/2}\right)\Big|_0^1$$

$$\boxed{\begin{array}{lcl} dv = \sqrt{x+2}\,dx & \Rightarrow & v = \frac{2}{3}(x+2)^{\frac{3}{2}} \\ u = x & & du = dx \end{array}}$$

$$= \frac{2}{3}3^{\frac{3}{2}} - \left(\frac{2}{3}\frac{2}{5}3^{5/2} - \frac{2}{3}\frac{2}{5}2^{5/2}\right)$$

Example: Find $\int x^2 e^x\,dx$

Solution:

$$\int \underbrace{x^2}_{u}\underbrace{e^x\,dx}_{dv} = x^2\cdot e^x - \int \underbrace{e^x\,dx}_{dw}\,\underbrace{2x}_{z} = x^2\cdot e^x - \left(2xe^x - \int e^x\cdot 2\,dx\right)$$

$$\boxed{\begin{array}{lcl} u = x^2 & & du = 2x\,dx \\ dv = e^x\,dx & \Rightarrow & v = e^x \end{array}}$$

$$\boxed{\begin{array}{lcl} z = 2x & & dz = 2dx \\ dw = e^x dx & \Rightarrow & w = e^x \end{array}}$$

$$= x^2 e^x - 2xe^x + 2\int e^x dx$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

Example: Find $\int x^3\cdot e^{x^2}\,dx$

Solution attempt: $\int \underbrace{x^3}_{u}\,\underbrace{e^{x^2}\,dx}_{dv}$

$$\boxed{\begin{array}{ll} u = x^3 & du = 3x^2 dx \\ dv = e^{x^2} dx & v = ?\ \ !! \end{array}}$$

The LIATE / LAPTU "rule" doesn't work.

Solution: $\int x^3 e^{x^2}dx = \int \underbrace{x^2}_{u}\ \underbrace{x e^{x^2}dx}_{dv} = x^2 \cdot \frac{1}{2} e^{x^2} - \int \cancel{\frac{1}{2}} e^{x^2} \cdot \cancel{2}x\, dx = \frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + C$

$u = x^2 \quad du = 2x\,dx$

$dv = x e^{x^2}dx \quad v = \frac{1}{2} e^{x^2}$

$\boxed{\text{Find } \int x e^{x^2}dx \text{ by the substitution } w = x^2}$

- $\int \ln x\, dx = \int \underbrace{\ln x}_{u} \cdot \underbrace{1\, dx}_{dv} = x \ln x - \int \underbrace{x}_{v} \cdot \underbrace{\frac{1}{x}dx}_{du} = x \ln x - \int 1\, dx = x \ln x - x + C$

- $\int \arctan x\, dx = \int \underbrace{\arctan x}_{u} \cdot \underbrace{1\, dx}_{dv} = x \arctan x - \int \underbrace{x}_{v} \cdot \underbrace{\frac{1}{1+x^2}dx}_{du} = x \arctan x - \int \frac{1}{2} \frac{dw}{w}$

$\boxed{\begin{array}{c} 1+x^2 = w \\ 2x\,dx = dw \end{array}}$

$= x \arctan x - \frac{1}{2} \ln |w| + C$

$= x \arctan x - \frac{1}{2} \ln |1+x^2| + C$

- $\int \arcsin x\, dx = \int \arcsin x \cdot 1\, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx = x \arcsin x + \sqrt{1-x^2} + C$

Example: Find $\int_0^{\pi} 2^x \sin x\, dx$

Solution: $\int_0^{\pi} \underbrace{\sin x}_{u}\ \underbrace{2^x dx}_{dv} = \sin x \cdot \frac{2^x}{\ln 2} \Big|_0^{\pi} - \int_0^{\pi} \frac{2^x}{\ln 2} \cos x\, dx = 0 - \frac{1}{\ln 2} \int_0^{\pi} \overbrace{2^x \underbrace{\cos x}_{z}\, dx}^{dw}$

$u = \sin x \quad du = \cos x\, dx$

$dv = 2^x dx \quad v = \frac{2^x}{\ln 2}$

$\boxed{\int a^x dx = \frac{a^x}{\ln a} + C}$

$= \frac{-1}{\ln 2} \left(\frac{2^x}{\ln 2} \cos x \Big|_0^{\pi} - \int_0^{\pi} \frac{2^x}{\ln 2} (-\sin x)\, dx \right)$

$= \frac{-1}{\ln 2} \left(\frac{-2^{\pi}}{\ln 2} - \frac{1}{\ln 2} + \frac{1}{\ln 2} \int_0^{\pi} 2^x \sin x\, dx \right)$

So $\underbrace{\int_0^{\pi} 2^x \sin x\, dx}_{I} = \frac{2^{\pi}}{\ln^2 2} + \frac{1}{\ln^2 2} - \frac{1}{\ln^2 2} \underbrace{\int_0^{\pi} 2^x \sin x\, dx}_{I}$

$$\left(1+\frac{1}{\ln^2 2}\right)\int_0^{\pi} 2^x \sin x \, dx = \frac{2^{\pi}+1}{\ln^2 2} \quad \text{and so} \quad \int_0^{\pi} 2^x \sin x\, dx = \frac{\frac{2^{\pi}+1}{\ln^2 2}}{1+\frac{1}{\ln^2 2}}$$

- $$\underbrace{\int \sec^3 x\, dx}_{I} = \int \underbrace{\sec x}_{u}\, \underbrace{\sec^2 x\, dx}_{dv} = \sec x \tan x - \int \underbrace{\tan x}_{v}\, \underbrace{\sec x \tan x\, dx}_{du}$$

$$= \sec x \tan x - \int \overbrace{(\sec^2 x - 1)}^{\tan^2 x} \sec x \, dx$$

$$= \sec x \tan x - \underbrace{\int \sec^3 x\, dx}_{I} + \int \sec x\, dx$$

and so

$$I = \int \sec^2 x\, dx = \frac{\sec x \tan x + \ln|\sec x + \tan x|}{2} + C$$

Examples: Find $\int \ln(\sin(x^2)) \sin(2x^2)\, x\, dx$

Solution: $\int \ln(\sin(x^2)) \sin(2x^2)\, x\, dx = \int \ln(\sin(x^2))\, 2\sin(x^2) \overbrace{\cos(x^2)\, x\, dx}^{dw}$

$$\boxed{\begin{aligned} w &= \sin(x^2) \\ dw &= \cos(x^2) . 2x\, dx \end{aligned}}$$

$$= \int \underbrace{\ln(w)}_{u}\, \underbrace{w\, dw}_{dv} = \ln(w)\frac{w^2}{2} - \int \underbrace{\frac{w^2}{2}}_{v}\, \underbrace{\frac{1}{w}\, dw}_{du} = \ln(w)\frac{w^2}{2} - \int \frac{w}{2}\, dw$$

$$= \ln(w)\frac{w^2}{2} - \frac{w^2}{4} + C$$

$$= \ln(\sin(x^2))\frac{\sin^2(x^2)}{2} - \frac{\sin^2(x^2)}{4} + C$$

Example: Find $\int_1^2 x \arcsin\left(\frac{1}{x}\right) dx$.

Solution: $$\int_1^2 \overbrace{x\, \underbrace{\arcsin\left(\frac{1}{x}\right)}_{u} dx}^{dv} = \arcsin\left(\frac{1}{x}\right)\frac{x^2}{2}\Bigg|_1^2 - \int_1^2 \frac{\cancel{x^2}}{2}\frac{1}{\sqrt{1-\frac{1}{x^2}}} \cdot \frac{-1}{\cancel{x^2}}\, dx$$

$$= \left(\arcsin\left(\frac{1}{2}\right) . 2 - \arcsin(1) . \frac{1}{2}\right) + \int_1^2 \frac{1}{\sqrt{\frac{x^2-1}{x^2}}}\, dx$$

$$= \left(\frac{\pi}{3}\cdot 2 - \frac{\pi}{2}\cdot\frac{1}{2}\right) + \int_1^2 \frac{|x|}{\sqrt{x^2-1}}\,dx$$

$$= \frac{2\pi}{3} - \frac{\pi}{4} + \int_1^2 \frac{x}{\sqrt{x^2-1}}\,dx = \frac{5\pi}{12} + \int_0^3 \frac{1}{2}\frac{dw}{\sqrt{w}} = \frac{5\pi}{12} + \sqrt{w}\Big|_0^3$$

$x^2-1=w$

$2x\,dx = dw$

$$= \frac{5\pi}{12} + \sqrt{3} - \sqrt{0} = \frac{5\pi}{12} + \sqrt{3}$$

Let $I_n = \int_0^{\pi/2} \cos^n x\,dx$ for integers $n \geq 0$. Then

$I_0 = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}$ and $I_1 = \int_0^{\frac{\pi}{2}} \cos x\,dx = \sin x\Big|_0^{\frac{\pi}{2}} = 1-0=1$

and, by integration by parts, we get, for $n \geq 2$,

this part is for fun and not included as a topic that you're responsible

$$I_n = \int_0^{\pi/2} \cos^n x\,dx = \int_0^{\pi/2} \underbrace{\cos^{n-1} x}_{u}\ \underbrace{\cos x\,dx}_{dv}$$

$$= \cos^{n-1}x \sin x\Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x\,(n-1)\cos^{n-2}x\cdot(-\sin x)\,dx$$

$$= (0-0) + (n-1)\int_0^{\pi/2} \cos^{n-2}x \sin^2 x\,dx$$

$$= (n-1)\int_0^{\pi/2} \cos^{n-2}x\,(1-\cos^2 x)\,dx = (n-1)\int_0^{\pi/2}\left(\cos^{n-2}x - \cos^n x\right)dx$$

$$= (n-1)\left(\int_0^{\pi/2} \cos^{n-2}x\,dx - \int_0^{\frac{\pi}{2}} \cos^n x\,dx\right)$$

So $I_n = (n-1)(I_{n-2} - I_n)$ and

$$\boxed{I_n = \left(\frac{n-1}{n}\right) I_{n-2}}$$

Thus $I_0 = \frac{\pi}{2}$, $I_2 = \frac{1}{2}\underbrace{\frac{\pi}{2}}_{I_0}$, $I_4 = \frac{3}{4}\underbrace{\frac{1}{2}\frac{\pi}{2}}_{I_2}$, $I_6 = \frac{5}{6}\underbrace{\frac{3}{4}\frac{1}{2}\frac{\pi}{2}}_{I_4}$, ...

$I_1 = 1$, $I_3 = \frac{2}{3}\underbrace{1}_{I_1}$, $I_5 = \frac{4}{5}\underbrace{\frac{2}{3}\cdot 1}_{I_3}$, $I_7 = \frac{6}{7}\underbrace{\frac{4}{5}\frac{2}{3}\cdot 1}_{I_5}$, ...

Since $0 \leq \cos x \leq 1$ for $0 \leq x \leq \frac{\pi}{2}$, we have that

$$\underbrace{\int_0^{\pi/2} \cos^n x \cos^2 x \, dx}_{I_{2n+2}} \leq \underbrace{\int_0^{\frac{\pi}{2}} \cos^2 x \cos x \, dx}_{I_{2n+1}} \leq \underbrace{\int_0^{\frac{\pi}{2}} \cos^2 x \, dx}_{I_{2n}}$$

It follows that

$$\frac{I_{2n+2}}{I_{2n}} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1$$

and

$\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$, by the squeeze theorem, as

$$\frac{2n+1}{2n+2} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1$$

and $\lim_{n\to\infty} \frac{2n+1}{2n+2} = \lim_{n\to\infty} 1 = 1$

Thus

$$\lim_{n\to\infty} \frac{\frac{2n}{2n+1} \frac{(2n-2)}{(2n-1)} \frac{(2n-4)}{(2n-3)} \cdots \frac{4}{5} \frac{2}{3} \cdot 1}{\frac{(2n-1)}{2n} \frac{(2n-3)}{(2n-2)} \cdots \frac{3}{4} \frac{1}{2} \frac{\pi}{2}} = 1$$

and hence,

$$\lim_{n\to\infty} \frac{2n \cdot 2n \cdot (2n-2)(2n-2) \cdots 44 \cdot 22}{(2n+1)(2n-1)(2n-1) \cdots 55 \cdot 33 \cdot 1} = \frac{\pi}{2}$$

. So we get the Wallis formula

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdots}$$